STRUCTURE EQUATIONS ON GENERALIZED FINSLER MANIFOLDS

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Abstract. In this paper we generalize the classical structure equations of Riemannian geometry to generalized Finsler manifolds.

1 Introduction

In this paper we deduce structure equations on a manifold which is endowed with a generalized Finsler metric and an Ehresmann connection. In Riemannian geometry, the classical structure equations were adopted by Élie Cartan. However Cartan's formalism was hard to understand for the next generations. In the pull-back formalism of Finsler geometry used by us, it causes a problem that in Grassmann algebra of forms along projection $\tau\colon TM \to M$ we do not have the classical exterior derivative. The vertical and horizontal derivatives, which substitute for exterior derivative, were introduced in 1992 ([8], [14]), and these help us to generalize the structure equations. By using the index-free calculus, it turns out that out of the five partial torsions introduced by Makoto Matsumoto in Finsler geometry only two ones have 'real' torsion property ([7] Chapter II.10, Lemma 3.1).

2 Preliminaries

We follow the notation and conventions of [14] and [6] as far as feasible. However, for the readers' convenience, in this section we fix some terminology and recall some basic facts.

Throughout this paper, we use the Einstein summation convention. 'Manifold' will always mean a connected, second countable, Hausdorff, smooth manifold of dimension n, $n \geq 1$. If M is a manifold, $C^\infty(M)$ will denote the ring of smooth functions on M. The tangent bundle of M is $\tau\colon TM \to M$, while $\mathring{\tau}\colon \mathring{T}M \to M$ denotes the slit tangent bundle, where $\mathring{T}M$ stands for the set of nonzero tangent vectors to M.

The vertical lift of a function $f \in C^\infty(M)$ is $f^{\mathsf{v}} := f \circ \tau$, the complete lift $f^{\mathsf{c}} \in C^\infty(TM)$ of f is defined by $f^{\mathsf{c}}(v) := v(f)$, $v \in TM$.

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$\mathfrak{X}(M)$ denotes the $C^\infty(M)$-module of smooth vector fields on M. Any vector field X on M gives rise canonically two vector fields on TM, the vertical lift X^{v} of X and the complete lift X^{c} of X, determined by $X^{\mathsf{v}}f^{\mathsf{c}} = (Xf)^{\mathsf{v}}$, $X^{\mathsf{v}}f^{\mathsf{v}} = 0$ and $X^{\mathsf{c}}f^{\mathsf{c}} = (Xf)^{\mathsf{c}}$, $X^{\mathsf{c}}f^{\mathsf{v}} = (Xf)^{\mathsf{v}}$; $f \in C^\infty(M)$.

Let $\mathcal{A}^k(M)$ be C^∞-module of k-forms on M. Then $\mathcal{A}(M) := \oplus_{k=0}^n \mathcal{A}^k(M)$ is a graded algebra over $C^\infty(M)$, with multiplication given by the wedge product $\wedge$.
If $f \in C^\infty(M)$ then the one-form $\mathrm{d}f$ given by $\mathrm{d}f(X) = Xf$ $(X \in \mathfrak{X}(M))$ is the *differential* of f.

Let $\tau^*TM := TM \times_M TM := \{(u,v) \in TM \times TM \mid \tau(u) = \tau(v)\}$, and let $\tau^*\tau(u,v) := u$ for $(u,v) \in \tau^*TM$. Then $\tau^*\tau$ is a vector bundle with total space τ^*TM and base space TM, the *pull-back* of $\tau\colon TM \to M$ over τ. The $C^\infty(TM)$-module of sections of $\tau^*\tau$ will be denoted by $\mathrm{Sec}(\tau^*\tau)$. Any vector field X on M determines a smooth section

$$\widehat{X}\colon v \in TM \longmapsto (v, X \circ \tau(v)) \in TM \times_M TM \ ,$$

called the *basic section* associated to X.The $C^\infty(TM)$-module $\mathrm{Sec}(\tau^*\tau)$ is generated by the basic sections. Generic sections in $\mathrm{Sec}(\tau^*\tau)$ will be denoted by $\widetilde{X}, \widetilde{Y}, \dots$.

The dual of $\mathrm{Sec}(\tau^*\tau)$ will be denoted by $\mathcal{A}^1(\tau^*\tau)$, and its elements is called *one-forms along* τ. $\mathcal{A}(\tau^*\tau)$ is the Grassmann algebra of differential forms along τ.

Starting from the slit tangent bundle $\mathring{\tau}\colon \mathring{T}M \to M$, the pull-back bundle $\mathring{\tau}^*\tau\colon \mathring{T}M \times_M TM \to TM$ is constructed in the same way. Omitting the routine details, we remark that $\mathrm{Sec}(\tau^*\tau)$ may naturally be embedded into the $C^\infty(\mathring{T}M)$-module $\mathrm{Sec}(\mathring{\tau}^*\tau)$.

There exists a canonical injective bundle map $\mathbf{i}\colon TM \times_M TM \to TTM$ given by

$$\mathbf{i}(u,v) := \dot{c}(0) \ , \qquad \text{if} \quad c(t) := u + tv \quad (t \in \mathbb{R}) \ ,$$

and a canonical surjective bundle map

$$\mathbf{j}\colon TTM \to TM \times_M TM \ ,$$
$$w \in T_vTM \longmapsto \mathbf{j}(w) := (v, \tau_*(w)) \in \{v\} \times T_{\tau(v)}M \ .$$

Then $\mathbf{j} \circ \mathbf{i} = 0$. However, while $\mathbf{J} := \mathbf{i} \circ \mathbf{j}$ is a further important canonical object, the *vertical endomorphism* of TTM. The bundle maps $\mathbf{i}$ and $\mathbf{j}$ induce the tensorial maps (denoted by the same symbols)

$$\widetilde{X} \in \mathrm{Sec}(\tau^*\tau) \longmapsto \mathbf{i}\widetilde{X} := \mathbf{i} \circ \widetilde{X} \in \mathfrak{X}(TM) \qquad \text{and}$$
$$\xi \in \mathfrak{X}(TM) \longmapsto \mathbf{j}\xi := \mathbf{j} \circ \xi \in \mathrm{Sec}(\tau^*\tau) \ ,$$

so $\mathbf{J}$ may also be interpreted as a $C^\infty(TM)$-linear endomorphism of $\mathfrak{X}(TM)$. $\mathfrak{X}^{\mathsf{v}}(TM) := \mathbf{i}\mathrm{Sec}(\tau^*\tau)$ is the module of *vertical vector fields* on TM. The vertical vector fields form a subalgebra of the Lie algebra $\mathfrak{X}(TM)$ at the same time. For any vector field X on M we have $\mathbf{i}\widehat{X} = X^{\mathsf{v}}$ and $\mathbf{j}X^{\mathsf{c}} = \widehat{X}$.

An *Ehresmann connection* $\mathcal{H}$ over a manifold M is a right splitting of the canonical exact sequence

$$0 \longrightarrow TM \times_M TM \xrightarrow{\mathbf{i}} TTM \xrightarrow{\mathbf{j}} TM \times_M TM \longrightarrow 0 \ ,$$

which is smooth only on $\mathring{T}M \times_M TM$, and given on $o(M) \times_M TM$ by $\mathcal{H}(o(p), v) := (o_*)_p(v)$; $p \in M$, $v \in T_pM$, where $o \in \mathfrak{X}(M)$ is the zero vector field. We associate to any Ehresmann connection $\mathcal{H}$ the *horizontal projector* $\mathbf{h} := \mathcal{H} \circ \mathbf{j}$, the *vertical projector* $\mathbf{v} = 1_{TTM} - \mathbf{h}$ and the *vertical map* $\mathcal{V} := \mathbf{i}^{-1} \circ \mathbf{v}$. The *horizontal lift* of a vector field $X \in \mathfrak{X}(M)$ with respect to $\mathcal{H}$ is $X^{\mathsf{h}} := \mathcal{H}(\widehat{X}) = \mathbf{h}X^{\mathsf{c}} \in \mathfrak{X}(\mathring{T}M)$.

The map $\ell^{\mathsf{h}} \colon X \in \mathfrak{X}(M) \longmapsto \ell^{\mathsf{h}}(X) := X^{\mathsf{h}}$ is said to be the horizontal lifting with respect to $\mathcal{H}$.

An Ehresmann connection $\mathcal{H}$ determines a covariant derivative operator ∇ in the pull-back bundle $\tau^*\tau$ by the rule

$$\nabla_\xi \widetilde{Y} := \mathbf{j}[\mathbf{v}\xi, \mathcal{H}\widetilde{Y}] + \mathcal{V}[\mathbf{h}\xi, \mathbf{i}\widetilde{Y}] \ ; \qquad \xi \in \mathfrak{X}(TM), \widetilde{Y} \in \mathrm{Sec}(\tau^*\tau) \ .$$

∇ is said to be the *Berwald derivative* induced by $\mathcal{H}$. Its *v-part* ∇^{v} and *h-part* ∇^{h} are defined by

$$\nabla^{\mathsf{v}}_{\widetilde{X}} \widetilde{Y} := \nabla_{\mathbf{i}\widetilde{X}} \widetilde{Y} = \mathbf{j}[\mathbf{i}\widetilde{X}, \mathcal{H}\widetilde{Y}]$$

and

$$\nabla^{\mathsf{h}}_{\widetilde{X}} \widetilde{Y} := \nabla_{\mathcal{H}\widetilde{X}} \widetilde{Y} = \mathcal{V}[\mathcal{H}\widetilde{X}, \mathbf{i}\widetilde{Y}]$$

$(\widetilde{X}, \widetilde{Y} \in \mathrm{Sec}(\tau^*\tau))$. If X and Y are vector fields on M, then $\nabla^{\mathsf{v}}_{\widehat{X}} \widehat{Y} = 0$ and $\mathbf{i}\nabla^{\mathsf{h}}_{\widehat{X}} \widehat{Y} = \left[X^{\mathsf{h}}, Y^{\mathsf{v}}\right]$.
The importance of the Berwald derivative lies, among others, in the fact that the basic geometric data (torsions, curvature, etc.) of an Ehresmann connection $\mathcal{H}$ may conveniently be defined in terms of the Berwald derivative induced by $\mathcal{H}$. In this paper we need the following $(\widetilde{X}, \widetilde{Y} \in \mathrm{Sec}(\tau^*\tau))$:

$$\mathbf{T}(\widetilde{X}, \widetilde{Y}) := \nabla^{\mathsf{h}}_{\widetilde{X}} \widetilde{Y} - \nabla^{\mathsf{h}}_{\widetilde{Y}} \widetilde{X} - \mathbf{j}[\mathcal{H}\widetilde{X}, \mathcal{H}\widetilde{Y}] \quad - \quad \text{the } \textit{torsion} \text{ of } \mathcal{H}, \tag{1}$$

$$\mathbf{R}(\widetilde{X}, \widetilde{Y}) := -\mathcal{V}[\mathcal{H}\widetilde{X}, \mathcal{H}\widetilde{Y}] \quad - \quad \text{the } \textit{curvature} \text{ of } \mathcal{H}. \tag{2}$$

3 Generalized Finsler manifolds and torsions of a Finsler connection

As in general, by *covariant derivative operator* in the vector bundle $\tau^*\tau$ we mean an $\mathbb{R}$-bilinear map

$$D \colon (\xi, \widetilde{X}) \in \mathfrak{X}(TM) \times \mathrm{Sec}(\tau^*\tau) \longmapsto D_\xi \widetilde{X} \in \mathrm{Sec}(\tau^*\tau)$$

which is tensorial in its first variable and derivation in its second variable.

The *curvature of* D is the

$$R^D(\xi,\eta)\widetilde{X} := D_\xi D_\eta \widetilde{X} - D_\eta D_\xi \widetilde{X} - D_{[\xi,\eta]}\widetilde{X} \tag{3}$$

$C^\infty(TM)$-trilinear map.

A *pseudo-Riemannian metric* on $\tau^*\tau$ is a mapping g that sends a non-degenerate symmetric bilinear form

$$g_v\colon \big(\{v\}\times T_{\tau(v)}M\big)\times\big(\{v\}\times T_{\tau(v)}M\big)\longrightarrow \mathbb{R}$$

(or simply $g_v\colon T_{\tau(v)}M\times T_{\tau(v)}M\to\mathbb{R}$) to every vector $v\in \mathring{T}M$ such that the function

$$g(\widetilde{X},\widetilde{Y})\colon \mathring{T}M\to\mathbb{R},\quad v\longmapsto g(\widetilde{X},\widetilde{Y})(v) := g_v(\widetilde{X}(v),\widetilde{Y}(v))$$

is smooth for any two sections $\widetilde{X},\widetilde{Y}\in\mathrm{Sec}(\mathring{\tau}^*\tau)$.

The pair (M,g) is said to be a *generalized Finsler manifold*, if g is a pseudo-Riemannian metric in $\tau^*\tau$. Then we also say that g is a *generalized metric*.

A covariant derivative operator $D\colon \mathfrak{X}(TM)\times\mathrm{Sec}(\tau^*\tau)\to\mathrm{Sec}(\tau^*\tau)$ in (M,g) is said to be *metric* if

$$D_\xi g(\widetilde{X},\widetilde{Y}) = \xi g(\widetilde{X},\widetilde{Y}) - g(D_\xi\widetilde{X},\widetilde{Y}) - g(\widetilde{X},D_\xi\widetilde{Y}) = 0\ . \tag{4}$$

Let $\mathcal{H}$ be an Ehresmann connection over M and let D be a covariant derivative operator in $\tau^*\tau$. Then the pair $(D,\mathcal{H})$ is called a *Finsler connection*. By the *torsion* of D we mean the map

$$T^D(\xi,\eta) := D_\xi\mathbf{j}\eta - D_\eta\mathbf{j}\xi - \mathbf{j}[\xi,\eta]\ ,\qquad (\xi,\eta\in\mathfrak{X}(TM))\ .$$

By the $\mathcal{V}$*-torsion* of D we mean the map

$$T^D_{\mathcal{V}}(\xi,\eta) := D_\xi\mathcal{V}\eta - D_\eta\mathcal{V}\xi - \mathcal{V}[\xi,\eta]\ ,\qquad (\xi,\eta\in\mathfrak{X}(TM))\ .$$

It is easy to see that T^D and $T^D_{\mathcal{V}}$ are tensor fields.

We define the following five 'partial torsions' which are introduced by M. Matsumoto ([7] Chapter II.10):

$$\mathcal{T}(\widetilde{X},\widetilde{Y}) := T^D(\mathcal{H}\widetilde{X},\mathcal{H}\widetilde{Y})\qquad \text{h-horizontal torsion,} \tag{5}$$

$$\mathcal{S}(\widetilde{X},\widetilde{Y}) := T^D(\mathcal{H}\widetilde{X},\mathbf{i}\widetilde{Y})\qquad \text{h-mixed torsion/Finsler torsion,} \tag{6}$$

$$\mathbf{R}^1(\widetilde{X},\widetilde{Y}) := T^D_{\mathcal{V}}(\mathcal{H}\widetilde{X},\mathcal{H}\widetilde{Y})\qquad \text{v-horizontal torsion,} \tag{7}$$

$$\mathbf{P}^1(\widetilde{X},\widetilde{Y}) := T^D_{\mathcal{V}}(\mathcal{H}\widetilde{X},\mathbf{i}\widetilde{Y})\qquad \text{v-mixed torsion,} \tag{8}$$

$$\mathbf{Q}^1(\widetilde{X},\widetilde{Y}) := T^D_{\mathcal{V}}(\mathbf{i}\widetilde{X},\mathbf{i}\widetilde{Y})\qquad \text{v-vertical torsion;} \tag{9}$$

$(\widetilde{X},\widetilde{Y}\in\mathrm{Sec}(\tau^*\tau))$.

The following formulae can be obtained by a straightforward calculation.

Lemma 3.1 *Let $(D, \mathcal{H})$ be a Finsler connection over M and let ∇ be the Berwald derivative induced by $\mathcal{H}$. Then for every $\widetilde{X}, \widetilde{Y} \in \mathrm{Sec}(\tau^*\tau)$*

$$\begin{aligned}
\mathcal{T}(\widetilde{X}, \widetilde{Y}) =& D_{\mathcal{H}\widetilde{X}}\widetilde{Y} - D_{\mathcal{H}\widetilde{Y}}\widetilde{X} - \mathbf{j}[\mathcal{H}\widetilde{X}, \mathcal{H}\widetilde{Y}] \ , && (10)\\
\mathcal{S}(\widetilde{X}, \widetilde{Y}) =& \nabla_{\mathbf{i}\widetilde{Y}}\widetilde{X} - D_{\mathbf{i}\widetilde{Y}}\widetilde{X} \ , && (11)\\
\mathbf{R}^1(\widetilde{X}, \widetilde{Y}) =& \mathbf{R}(\widetilde{X}, \widetilde{Y}) \ , && (12)\\
\mathbf{P}^1(\widetilde{X}, \widetilde{Y}) =& D_{\mathcal{H}\widetilde{X}}\widetilde{Y} - \nabla_{\mathcal{H}\widetilde{X}}\widetilde{Y} \ , && (13)\\
\mathbf{Q}^1(\widetilde{X}, \widetilde{Y}) =& D_{\mathbf{i}\widetilde{X}}\widetilde{Y} - D_{\mathbf{i}\widetilde{Y}}\widetilde{X} - \mathbf{i}^{-1}[\mathbf{i}\widetilde{X}, \mathbf{i}\widetilde{Y}] \ . && (14)
\end{aligned}$$

We have an important remark that among the above mentioned five partial torsions only two ones have 'real' torsion property: the h-horizontal torsion $\mathcal{T}$ and the v-vertical torsion $\mathbf{Q}^1$.

Proposition 3.2 *Let (M, g) be a generalized Finsler manifold endowed with an Ehresmann connection $\mathcal{H}$. Then exists a unique covariant derivative operator D such that*
(i) *D is metric,*
(ii) $\mathcal{T}(\widetilde{X}, \widetilde{Y}) = \mathbf{T}(\widetilde{X}, \widetilde{Y})$,
(iii) $\mathbf{Q}^1(\widetilde{X}, \widetilde{Y}) = 0$,
for any $\widetilde{X}, \widetilde{Y} \in \mathrm{Sec}(\tau^*\tau)$.

For a proof we refer to [6].
We say that D is the *canonical covariant derivative* for the structure $(M, g, \mathcal{H})$.

4 Structure equations

The following concepts and results can be found in [14] Chapter 2, Section E.

Lemma and Definition 4.1 *There is a unique graded derivation $\mathrm{d}^{\mathsf{v}} \colon \mathcal{A}(\tau^*\tau) \to \mathcal{A}(\tau^*\tau)$ of degree 1 such that*

$$\begin{aligned}
(\mathrm{d}^{\mathsf{v}} f)(\widetilde{X}) :=& \mathrm{d}f(\mathbf{i}\widetilde{X}) \ , \qquad \textit{and}\\
\mathrm{d}^{\mathsf{v}}\widetilde{\alpha}(\widetilde{X}_1, \dots, \widetilde{X}_{k+1}) :=& \sum_{i=1}^{k+1} (-1)^{i+1} (\mathbf{i}\widetilde{X}_i)\widetilde{\alpha}(\widetilde{X}_1, \dots, \hat{\widetilde{X}}_i, \dots, \widetilde{X}_{k+1}) +\\
&+ \sum_{1 \le i < j \le k+1} (-1)^{i+j} \widetilde{\alpha}(\mathbf{i}^{-1}[\mathbf{i}\widetilde{X}_i, \mathbf{i}\widetilde{X}_j], \dots, \hat{\widetilde{X}}_i, \dots, \hat{\widetilde{X}}_j, \dots, \widetilde{X}_{k+1})
\end{aligned}$$

for all $f \in C^\infty(TM)$, $\widetilde{X}, \widetilde{X}_i \in \mathrm{Sec}(\tau^*\tau)$ $(i = 1, \dots, k+1)$ and $\widetilde{\alpha} \in \mathcal{A}^k(\tau)$. d^{v} is said to be the vertical exterior derivative *on $\mathcal{A}(\tau^*\tau)$.*

Lemma and Definition 4.2 *Let $\mathcal{H}$ be an Ehresmann connection. There is a unique graded derivation* $\mathrm{d}^{\mathsf{h}}\colon \mathcal{A}(\tau^*\tau) \to \mathcal{A}(\tau^*\tau)$ *of degree 1 such that*

$$(\mathrm{d}^{\mathsf{h}} f)(\widetilde{X}) := \mathrm{d}f(\mathcal{H}\widetilde{X}) \ , \qquad \textit{and}$$

$$\mathrm{d}^{\mathsf{h}}\widetilde{\alpha}(\widetilde{X}_1, \dots, \widetilde{X}_{k+1}) := \sum_{i=1}^{k+1} (-1)^{i+1} (\mathcal{H}\widetilde{X}_i)\widetilde{\alpha}(\widetilde{X}_1, \dots, \hat{\widetilde{X}}_i, \dots, \widetilde{X}_{k+1}) + \\ + \sum_{1 \le i < j \le k+1} (-1)^{i+j} \widetilde{\alpha}(\mathbf{j}[\mathcal{H}\widetilde{X}_i, \mathcal{H}\widetilde{X}_j], \dots, \hat{\widetilde{X}}_i, \dots, \hat{\widetilde{X}}_j, \dots, \widetilde{X}_{k+1})$$

for all $f \in C^\infty(TM)$, $\widetilde{X}, \widetilde{X}_i \in \mathrm{Sec}(\tau^*\tau)$ $(i = 1, \dots, k+1)$ and $\widetilde{\alpha} \in \mathcal{A}^k(\tau)$. d^{h} is called the horizontal exterior derivative *on $\mathcal{A}(\tau^*\tau)$ with respect to $\mathcal{H}$.*

In the above formulas the notation $\hat{\widetilde{X}}$ means that the argument $\widetilde{X}$ is deleted.

If $k = 1$, we obtain

$$\mathrm{d}^{\mathsf{v}}\widetilde{\alpha}(\widetilde{X}_1, \widetilde{X}_2) = (\mathbf{i}\widetilde{X}_1)\widetilde{\alpha}(\widetilde{X}_2) - (\mathbf{i}\widetilde{X}_2)\widetilde{\alpha}(\widetilde{X}_1) - \widetilde{\alpha}(\mathcal{V}[\mathbf{i}\widetilde{X}_1, \mathbf{i}\widetilde{X}_2]) \ , \tag{15}$$

$$\mathrm{d}^{\mathsf{h}}\widetilde{\alpha}(\widetilde{X}_1, \widetilde{X}_2) = (\mathcal{H}\widetilde{X}_1)\widetilde{\alpha}(\widetilde{X}_2) - (\mathcal{H}\widetilde{X}_2)\widetilde{\alpha}(\widetilde{X}_1) - \widetilde{\alpha}(\mathbf{j}[\mathcal{H}\widetilde{X}_1, \mathcal{H}\widetilde{X}_2]) \ . \tag{16}$$

Let (M, g) be a generalized Finsler manifold. Let $(\widetilde{E}_i)_{i=1}^n$ be a family of g-orthonormal sections in $\mathrm{Sec}(\tau^*\tau)$ on open subset $\mathcal{U} \subset TM$:

$$\widetilde{E}_i\colon v \in \mathcal{U} \longmapsto \widetilde{E}_i(v) \in T_{\tau(v)}M \ ,$$
$$g(\widetilde{E}_i, \widetilde{E}_j) = \delta_{ij} \quad (1 \le i, j \le n) \ .$$

Let $(\widetilde{\Theta}^i)_{i=1}^n$ be denote the family of dual 1-forms of $(\widetilde{E}_i)_{i=1}^n$. Then

$$\widetilde{\Theta}^i(\widetilde{E}_j) = \delta^i_j \ , \quad 1 \le i, j \le n \ .$$

Using these local frame fields, every section $\widetilde{X}$ of $\overset{\circ}{\tau}{}^*\tau$ over $\mathcal{U}$ can be expressed as

$$\widetilde{X} = \widetilde{\Theta}^i(\widetilde{X})\widetilde{E}_i \ . \tag{17}$$

Indeed,

$$\widetilde{\Theta}^i(\widetilde{X})\widetilde{E}_i = \widetilde{\Theta}^i(\widetilde{X}^j\widetilde{E}_j)\widetilde{E}_i = \widetilde{X}^j\widetilde{\Theta}^i(\widetilde{E}_j)\widetilde{E}_i = \widetilde{X}^j\delta^i_j\widetilde{E}_i = \widetilde{X}^j\widetilde{E}_j = \widetilde{X} \ .$$

If $\mathcal{H}$ is an Ehresmann connection on M, then there exist 2-forms $\widetilde{\vartheta}^i$ along τ (on $\mathcal{U}$) such that

$$\mathbf{T}(\widetilde{X}, \widetilde{Y}) = \widetilde{\vartheta}^i(\widetilde{X}, \widetilde{Y})\widetilde{E}_i \ , \tag{18}$$

for any sections $\widetilde{X}, \widetilde{Y}$ of $\tau^*\tau$ over $\mathcal{U}$.

Let R^D be the curvature tensor of D. Then there exist 2-forms $\widetilde{\Omega}^i_j$ along τ such that

$$R^D(\xi, \eta)\widetilde{E}_j = \widetilde{\Omega}^i_j(\xi, \eta)\widetilde{E}_i \ . \tag{19}$$

We say that $\widetilde{\vartheta}^i$ are the *torsion two-forms*, $\widetilde{\Omega}^i_j$ are the *curvature two-forms* of the Ehresmann connection with respect to $(\widetilde{E}_i)_{i=1}^n$.

Theorem and Definition 4.3 *Let (M, g) be a generalized Finsler manifold. Let $\mathcal{H}$ be an Ehresmann connection and let D be the canonical covariant derivative for $(M, g, \mathcal{H})$. Suppose that g is positive definite and let $\mathcal{U}$ be an open subset of TM. Define $(\widetilde{E}_i)_{i=1}^n$ and $(\widetilde{\Theta}^i)_{i=1}^n$ as above. Then there exists a unique family $\left(\widetilde{\omega}^i_j\right)_{1\leq i,j\leq n}$ of 1-forms on $\mathcal{U}$ such that*

$$\widetilde{\omega}^i_j = -\widetilde{\omega}^j_i \ , \tag{20}$$

$$\mathrm{d}^{\mathsf{v}}\widetilde{\Theta}^i = -(\widetilde{\omega}^i_j \circ \mathbf{i}) \wedge \widetilde{\Theta}^j \qquad (1 \leq i \leq n), \tag{21}$$

$$\mathrm{d}^{\mathsf{h}}\widetilde{\Theta}^i = -(\widetilde{\omega}^i_j \circ \mathcal{H}) \wedge \widetilde{\Theta}^j - \widetilde{\vartheta}^i \qquad (1 \leq i \leq n), \tag{22}$$

$$\widetilde{\Omega}^i_j = d\widetilde{\omega}^i_j + \widetilde{\omega}^i_k \wedge \widetilde{\omega}^k_j \ . \tag{23}$$

The 1-forms $\widetilde{\omega}^i_j$ are said to be the connection forms*. Relations (21) and (22) are called the* first structure equations*. Relations (23) are mentioned as the* second structure equations.

Remark. Owing to 3.2, the structure equations of v-vertical torsion $\mathbf{Q}^1$ are not relevant.

Proof. Define the 1-forms $\widetilde{\omega}^i_j$ by

$$\widetilde{\omega}^i_j(\xi) := \widetilde{\Theta}^i(D_\xi \widetilde{E}_j) \qquad (\xi \in \mathfrak{X}(TM)) \ .$$

(1) Since D is metric, we have

$$\begin{aligned} 0 = (D_\xi g)(\widetilde{E}_i, \widetilde{E}_j) &= \xi g(\widetilde{E}_i, \widetilde{E}_j) - g(D_\xi \widetilde{E}_i, \widetilde{E}_j) - g(D_\xi \widetilde{E}_j, \widetilde{E}_i) \overset{(17)}{=} \\ &= \xi \delta_{ij} - g(\widetilde{\Theta}^k(D_\xi \widetilde{E}_i)\widetilde{E}_k, \widetilde{E}_j) - g(\widetilde{\Theta}^k(D_\xi \widetilde{E}_j)\widetilde{E}_k, \widetilde{E}_i) = \\ &= -g(\widetilde{\omega}^k_i \widetilde{E}_k, \widetilde{E}_j) - g(\widetilde{\omega}^k_j \widetilde{E}_k, \widetilde{E}_i) = \\ &= -\widetilde{\omega}^k_i g(\widetilde{E}_k, \widetilde{E}_j) - \widetilde{\omega}^k_j g(\widetilde{E}_k, \widetilde{E}_i) = -\widetilde{\omega}^j_i - \widetilde{\omega}^i_j \ , \end{aligned}$$

whence (20).

(2) *Equations (21).* The left-hand side of (21) can be manipulated as follows:

$$\begin{aligned} \mathrm{d}^{\mathsf{v}}\widetilde{\Theta}^i(\widetilde{E}_k, \widetilde{E}_l) &\overset{(15)}{=} (\mathbf{i}\widetilde{E}_k)\widetilde{\Theta}^i \widetilde{E}_l - (\mathbf{i}\widetilde{E}_l)\widetilde{\Theta}^i \widetilde{E}_k - \widetilde{\Theta}^i(\mathcal{V}[\mathbf{i}\widetilde{E}_k, \mathbf{i}\widetilde{E}_l]) = \\ &= (\mathbf{i}\widetilde{E}_k)\delta^i_l - (\mathbf{i}\widetilde{E}_l)\delta^i_k - \widetilde{\Theta}^i(\mathcal{V}[\mathbf{i}\widetilde{E}_k, \mathbf{i}\widetilde{E}_l]) = -\widetilde{\Theta}^i(\mathcal{V}[\mathbf{i}\widetilde{E}_k, \mathbf{i}\widetilde{E}_l]) \ . \end{aligned}$$

Evaluating the right-hand side at $\left(\widetilde{E}_k, \widetilde{E}_l\right)$ we find

$$\begin{aligned} ((\widetilde{\omega}^i_j \circ \mathbf{i}) \wedge \widetilde{\Theta}^j)(\widetilde{E}_k, \widetilde{E}_l) &= \widetilde{\omega}^i_j(\mathbf{i}\widetilde{E}_k)\widetilde{\Theta}^j \widetilde{E}_l - \widetilde{\omega}^i_j(\mathbf{i}\widetilde{E}_l)\widetilde{\Theta}^j \widetilde{E}_k = \\ &= \widetilde{\omega}^i_l(\mathbf{i}\widetilde{E}_k)\widetilde{\omega}^i_k(\mathbf{i}\widetilde{E}_l) = \widetilde{\Theta}^i(D_{\mathbf{i}\widetilde{E}_k}\widetilde{E}_l) - \widetilde{\Theta}^i(D_{\mathbf{i}\widetilde{E}_l}\widetilde{E}_k) = \\ &= \widetilde{\Theta}^i(D_{\mathbf{i}\widetilde{E}_k}\widetilde{E}_l - D_{\mathbf{i}\widetilde{E}_l}\widetilde{E}_k) = \widetilde{\Theta}^i(\mathcal{V}[\mathbf{i}\widetilde{E}_k, \mathbf{i}\widetilde{E}_l]) \ , \end{aligned}$$

taking into account in the last step that $\mathbf{Q}^1 = 0$ by Proposition 3.2, and hence $0 = \mathbf{Q}^1(\widetilde{E}_k, \widetilde{E}_l) = D_{\mathbf{i}\widetilde{E}_k}\widetilde{E}_l - D_{\mathbf{i}\widetilde{E}_l}\widetilde{E}_k - \mathcal{V}[\mathbf{i}\widetilde{E}_k, \mathbf{i}\widetilde{E}_l]$.

(3) *Equations (22).*

$$\begin{aligned}\mathrm{d}^{\mathsf{h}}\widetilde{\Theta}^i(\widetilde{E}_k, \widetilde{E}_l) &\overset{(16)}{=} (\mathcal{H}\widetilde{E}_k)\widetilde{\Theta}^i\widetilde{E}_l - (\mathcal{H}\widetilde{E}_l)\widetilde{\Theta}^i\widetilde{E}_k - \widetilde{\Theta}^i(\mathbf{j}[\mathcal{H}\widetilde{E}_k, \mathcal{H}\widetilde{E}_l]) = \\ &= (\mathcal{H}\widetilde{E}_k)\delta^i_l - (\mathcal{H}\widetilde{E}_l)\delta^i_k - \widetilde{\Theta}^i(\mathbf{j}[\mathcal{H}\widetilde{E}_k, \mathcal{H}\widetilde{E}_l]) = -\widetilde{\Theta}^i(\mathbf{j}[\mathcal{H}\widetilde{E}_k, \mathcal{H}\widetilde{E}_l])\end{aligned}$$

Since $\mathbf{T}(\widetilde{X}, \widetilde{Y}) \overset{3.2.(ii)}{=} D_{\mathcal{H}\widetilde{X}}\widetilde{Y} - D_{\mathcal{H}\widetilde{Y}}\widetilde{X} - \mathbf{j}[\mathcal{H}\widetilde{X}, \mathcal{H}\widetilde{Y}]$, we get

$$\begin{aligned}((\widetilde{\omega}^i_j \circ \mathcal{H}) \wedge \widetilde{\Theta}^j - \widetilde{\vartheta}^i)(\widetilde{E}_k, \widetilde{E}_l) &= \widetilde{\omega}^i_j(\mathcal{H}\widetilde{E}_k)\widetilde{\Theta}^j\widetilde{E}_l - \widetilde{\omega}^i_j(\mathcal{H}\widetilde{E}_l)\widetilde{\Theta}^j\widetilde{E}_k - \widetilde{\vartheta}^i(\widetilde{E}_k, \widetilde{E}_l) = \\ &= \widetilde{\omega}^i_l(\mathcal{H}\widetilde{E}_k) - \widetilde{\omega}^i_k(\mathcal{H}\widetilde{E}_l) - \widetilde{\vartheta}^i(\widetilde{E}_k, \widetilde{E}_l) = \\ &= \widetilde{\Theta}^i(D_{\mathcal{H}\widetilde{E}_k}\widetilde{E}_l) - \widetilde{\Theta}^i(D_{\mathcal{H}\widetilde{E}_l}\widetilde{E}_k) - \widetilde{\vartheta}^i(\widetilde{E}_k, \widetilde{E}_l) = \\ &= \widetilde{\Theta}^i(D_{\mathcal{H}\widetilde{E}_k}\widetilde{E}_l - D_{\mathcal{H}\widetilde{E}_l}\widetilde{E}_k) - \widetilde{\vartheta}^i(\widetilde{E}_k, \widetilde{E}_l) = \\ &= \widetilde{\Theta}^i(\mathbf{T}(\widetilde{E}_k, \widetilde{E}_l) + \mathbf{j}[\mathcal{H}\widetilde{E}_k, \mathcal{H}\widetilde{E}_l]) - \widetilde{\vartheta}^i(\widetilde{E}_k, \widetilde{E}_l) \overset{(18)}{=} \\ &= \widetilde{\Theta}^i(\widetilde{\vartheta}^s(\widetilde{E}_k, \widetilde{E}_l)\widetilde{E}_s) + \widetilde{\Theta}^i(\mathbf{j}[\mathcal{H}\widetilde{E}_k, \mathcal{H}\widetilde{E}_l]) - \widetilde{\vartheta}^i(\widetilde{E}_k, \widetilde{E}_l) = \\ &= \widetilde{\Theta}^i(\mathbf{j}[\mathcal{H}\widetilde{E}_k, \mathcal{H}\widetilde{E}_l]) \ .\end{aligned}$$

(4) *Equations (23).* By using the definition of D and R^D, relation (17), we find

$$\begin{aligned}\widetilde{\Omega}^i_j(\xi, \eta)\widetilde{E}_i &\overset{(19)}{=} R^D(\xi, \eta)\widetilde{E}_j = D_\xi D_\eta\widetilde{E}_j - D_\eta D_\xi\widetilde{E}_j - D_{[\xi,\eta]}\widetilde{E}_j = \\ &= D_\xi(\widetilde{\Theta}^k(D_\eta\widetilde{E}_j)\widetilde{E}_k) - D_\eta(\widetilde{\Theta}^k(D_\xi\widetilde{E}_j)\widetilde{E}_k) - \widetilde{\Theta}^i(D_{[\xi,\eta]}\widetilde{E}_j)\widetilde{E}_i = \\ &= \xi(\widetilde{\Theta}^k(D_\eta\widetilde{E}_j))\widetilde{E}_k + \widetilde{\Theta}^k(D_\eta\widetilde{E}_j)D_\xi\widetilde{E}_k - \\ &\quad - \eta(\widetilde{\Theta}^k(D_\xi\widetilde{E}_j))\widetilde{E}_k - \widetilde{\Theta}^k(D_\xi\widetilde{E}_j)D_\eta\widetilde{E}_k - \widetilde{\Theta}^i(D_{[\xi,\eta]}\widetilde{E}_j)\widetilde{E}_i = \\ &= \xi(\widetilde{\Theta}^i(D_\eta\widetilde{E}_j))\widetilde{E}_i - \eta(\widetilde{\Theta}^i(D_\xi\widetilde{E}_j))\widetilde{E}_i - \widetilde{\Theta}^i(D_{[\xi,\eta]}\widetilde{E}_j)\widetilde{E}_i + \\ &\quad + \widetilde{\omega}^k_j(\eta)D_\xi\widetilde{E}_k - \widetilde{\omega}^k_j(\xi)D_\eta\widetilde{E}_k = \\ &= \xi(\widetilde{\Theta}^i(D_\eta\widetilde{E}_j))\widetilde{E}_i - \eta(\widetilde{\Theta}^i(D_\xi\widetilde{E}_j))\widetilde{E}_i - \widetilde{\Theta}^i(D_{[\xi,\eta]}\widetilde{E}_j)\widetilde{E}_i + \\ &\quad + \widetilde{\omega}^k_j(\eta)\widetilde{\Theta}^i(D_\xi\widetilde{E}_k)\widetilde{E}_i - \widetilde{\omega}^k_j(\xi)\widetilde{\Theta}^i(D_\eta\widetilde{E}_k)\widetilde{E}_i = \\ &= \xi(\widetilde{\omega}^i_j(\eta))\widetilde{E}_i - \eta(\widetilde{\omega}^i_j(\xi))\widetilde{E}_i - \widetilde{\omega}^i_j([\xi,\eta])\widetilde{E}_i + \\ &\quad + \widetilde{\omega}^k_j(\eta)\widetilde{\omega}^i_k(\xi)\widetilde{E}_i - \widetilde{\omega}^k_j(\xi)\widetilde{\omega}^i_k(\eta)\widetilde{E}_i = \\ &= \left(\xi(\widetilde{\omega}^i_j(\eta)) - \eta(\widetilde{\omega}^i_j(\xi)) - \widetilde{\omega}^i_j([\xi,\eta]) + \widetilde{\omega}^i_k(\xi)\widetilde{\omega}^k_j(\eta) - \widetilde{\omega}^i_k(\eta)\widetilde{\omega}^k_j(\xi)\right)\widetilde{E}_i \ .\end{aligned}$$

On the other hand,

$$\begin{aligned}(\mathrm{d}\widetilde{\omega}^i_j + \widetilde{\omega}^i_k \wedge \widetilde{\omega}^k_j)(\xi, \eta) &= d\widetilde{\omega}^i_j(\xi, \eta) + \widetilde{\omega}^i_k(\xi)\widetilde{\omega}^k_j(\eta) - \widetilde{\omega}^i_k(\eta)\widetilde{\omega}^k_j(\xi) = \\ &= \xi(\widetilde{\omega}^i_j(\eta)) - \eta(\widetilde{\omega}^i_j(\xi)) - \widetilde{\omega}^i_j([\xi,\eta]) + \widetilde{\omega}^i_k(\xi)\widetilde{\omega}^k_j(\eta) - \widetilde{\omega}^i_k(\eta)\widetilde{\omega}^k_j(\xi) \ ,\end{aligned}$$

which concludes the proof of (23).

(5) *Uniqueness of the family* $(\widetilde{\omega}^i_j)$. We use the fact that any 1-form of an open subset of TM is completely determined by its action over vertical and horizontal vector fields.
First we prove that the effect of the connection forms on vertical vector fields is well-defined. We start on (21) and paragraph 2 of this proof.

$$\mathrm{d}^{\mathsf{v}}\widetilde{\Theta}^i(\widetilde{E}_j,\widetilde{E}_k) = \widetilde{\omega}^i_j(\mathbf{i}\widetilde{E}_k) - \widetilde{\omega}^i_k(\mathbf{i}\widetilde{E}_j) \ , \tag{24}$$
$$\mathrm{d}^{\mathsf{v}}\widetilde{\Theta}^j(\widetilde{E}_k,\widetilde{E}_i) = \widetilde{\omega}^j_k(\mathbf{i}\widetilde{E}_i) - \widetilde{\omega}^j_i(\mathbf{i}\widetilde{E}_k) \ , \tag{25}$$
$$\mathrm{d}^{\mathsf{v}}\widetilde{\Theta}^k(\widetilde{E}_i,\widetilde{E}_j) = \widetilde{\omega}^k_i(\mathbf{i}\widetilde{E}_j) - \widetilde{\omega}^k_j(\mathbf{i}\widetilde{E}_i) \ . \tag{26}$$

Now we add the first two equalities, and subtract the third. Taking into account (20), we obtain

$$\widetilde{\omega}^i_j(\mathbf{i}\widetilde{E}_k) = \frac{1}{2}\left(\mathrm{d}^{\mathsf{v}}\widetilde{\Theta}^i(\widetilde{E}_j,\widetilde{E}_k) + \mathrm{d}^{\mathsf{v}}\widetilde{\Theta}^j(\widetilde{E}_k,\widetilde{E}_i) - \mathrm{d}^{\mathsf{v}}\widetilde{\Theta}^k(\widetilde{E}_i,\widetilde{E}_j)\right) \ ,$$

and this relation proves the statement.
Similarly, we have

$$\mathrm{d}^{\mathsf{h}}\widetilde{\Theta}^i(\widetilde{E}_j,\widetilde{E}_k) = \widetilde{\omega}^i_j(\mathcal{H}\widetilde{E}_k) - \widetilde{\omega}^i_k(\mathcal{H}\widetilde{E}_j) + \widetilde{\vartheta}^i(\widetilde{E}_j,\widetilde{E}_k) \ , \tag{27}$$
$$\mathrm{d}^{\mathsf{h}}\widetilde{\Theta}^j(\widetilde{E}_k,\widetilde{E}_i) = \widetilde{\omega}^j_k(\mathcal{H}\widetilde{E}_i) - \widetilde{\omega}^j_i(\mathcal{H}\widetilde{E}_k) + \widetilde{\vartheta}^j(\widetilde{E}_k,\widetilde{E}_i) \ , \tag{28}$$
$$\mathrm{d}^{\mathsf{h}}\widetilde{\Theta}^k(\widetilde{E}_i,\widetilde{E}_j) = \widetilde{\omega}^k_i(\mathcal{H}\widetilde{E}_j) - \widetilde{\omega}^k_j(\mathcal{H}\widetilde{E}_i) + \widetilde{\vartheta}^k(\widetilde{E}_i,\widetilde{E}_j) \ . \tag{29}$$

Adding the first two equalities, and subtracting the third, by using (20) we find

$$\begin{aligned}\widetilde{\omega}^i_j(\mathcal{H}\widetilde{E}_k) =& \frac{1}{2}\left(\mathrm{d}^{\mathsf{h}}\widetilde{\omega}^i(\widetilde{E}_j,\widetilde{E}_k) + \mathrm{d}^{\mathsf{h}}\widetilde{\omega}^j(\widetilde{E}_k,\widetilde{E}_i) - \mathrm{d}^{\mathsf{h}}\widetilde{\omega}^k(\widetilde{E}_i,\widetilde{E}_j)\right) - \\ &- \frac{1}{2}\left(\widetilde{\vartheta}^i(\widetilde{E}_j,\widetilde{E}_k) + \widetilde{\vartheta}^j(\widetilde{E}_k,\widetilde{E}_i) - \widetilde{\vartheta}^k(\widetilde{E}_i,\widetilde{E}_j)\right) \ .\end{aligned}$$

□

References

[1] M. Crampin, On horizontal distributions on the tangent bundle of a differentiable manifold, *J. London Math. Soc., (2)* **3** (1971), 178–182.

[2] M. Crampin, Connections of Berwald type, *Publ. Math. Debrecen* **57** (2000), 455–473.

[3] M. de León and P. R. Rodrigues, *Methods of Differential Geometry in Analytical Mechanics*, North-Holland (Amsterdam, 1989).

[4] W. Greub, S. Halperin and R. Vanstone, *Connections, Curvature, and Cohomology*, Vol.I, Academic Press (New York, 1972).

[5] J. Grifone, Structure presque tangente et connexions I., *Ann. Inst. Fourier, Grenoble (1)* **22** (1972), 287–334.

[6] R. L. Lovas, J. Pék and J. Szilasi, Ehresmann connections, metrics and good metric derivatives, *Advanced Studies in Pure Mathematics*, 48 (2007), Finsler Geometry, Sapporo 2005 – In Memory of Makoto Matsumoto, 263–308.

[7] M. Matsumoto, *Foundations of Finsler Geometry and Special Finsler Spaces*, Kaiseisha Press (Otsu, 1986).

[8] E. Martinez, J. F. Cariñena and W. Sarlet, Derivations of differential forms along the tangent bundle projection, *Diff. Geom. Appl.* **2** (1992), 17–43.

[9] R. Miron, Metrical Finsler structures and metrical Finsler connections, *J. Math. Kyoto Univ.* **23** (1983), 219–224.

[10] T. Mestdag, J. Szilasi and V. Tóth, On the geometry of generalized metrics, *Publ. Math. Debrecen* **62** (2003), 511–545.

[11] A. Moór, Entwicklung einer Geometrie der allgemeinen metrischen Linienelementräume, *Acta Sci. Math. Szeged* **17** (1956), 85–120.

[12] A. Moór, Eine Verallgemeinerung der metrischen Übertragung in allgemeinen metrischen Räumen, *Publ. Math. Debrecen* **10** (1963), 145–150.

[13] P. Petersen, *Riemannian Geometry*, Springer (Berlin, 1968).

[14] J. Szilasi, *A Setting for Spray and Finsler Geometry*, In: P. L. Antonelli (ed.): Handbook of Finsler Geometry, Vol.2, Kluwer Academic Publishers (Dordrecht, 2003) 1182–1426.

[15] J. R. Vanstone, A generalization of Finsler Geometry, *Canad. J. Math.* **14** (1962), 87–112.